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Solvable multi-species reaction-diffusion processes, including the extended drop-push model

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Abstract. By considering the master equation of asymmetric exclusion process on a one-dimensional lattice, we obtain the most general boundary condition of the multi-species exclusion processes in which the number of particles is constant in time. This boundary condition introduces the various interactions to the particles, including ones which have been studied yet and the new ones. In these new models, the particles have simultaneously diffusion, the two-particle interactions $A_\alpha A_\beta \to A_\gamma A_\delta$, and the n-particle extended drop-push interaction. The constraints on reaction rates are obtained and in two-species case, they are solved to obtain a solvable model. The conditional probabilities of this model are calculated.

PACS. 82.20.Mj Nonequilibrium kinetics – 02.50.Ga Markov processes – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion

1 Introduction

One-dimensional asymmetric simple exclusion processes (ASEP) have been shown to be of physical interest in various problems in recent years. These problems are, for example: the kinetics of biopolymerization [1,2], traffic models [3], polymers in random media, dynamical models of interface growth [4,5], noisy Burgers equation [6], and study of the shocks [7,8]. There are many review articles in these fields, see for example [9–15].

The totally ASEP has been solved exactly in reference [16]. In this simple model, each lattice site is occupied by at most one particle and particles hop with rate one to their right-neighboring sites if they are not occupied. The model is completely specified by a master equation and a boundary condition, imposed on probabilities appear in the master equation. The coordinate Bethe ansatz has been used to show the factorization of the N-particle scattering matrix to the two-particle matrices.

By choosing other suitable boundary conditions, without changing the master equation, one may study the more complicated reaction-diffusion processes, even with long-range interaction. In reference [17], the so-called drop-push model has been studied by this method. In this model the particle hops to the next right site even it is occupied. The particle hops to this site by pushing all the neighboring particles to their next right sites, with a rate depending on the number of right neighboring particles. The generalization of this model, by considering both the right and left hopping, has been done in reference [18]. Further generalization of the ASEP boundary condition have been proposed in [19–21]. As a comprehensive review on the ASEP, see [22].

All of the above studies are about the single-species systems. If one considers the model with more-than-one species, the situation becomes more complicated. The source of complexity is the abovementioned factorization of N-particle scattering matrix, which in these cases restricts the two-particle S-matrices satisfies in some kind of spectral Yang-Baxter (SYB) equation. In reference [23], all the solvable two-species reaction-diffusion models, in which the number of particles is constant in time and the reaction rates are all equal, have been obtained. It was shown that there are 28 independent interactions (among 4096 possible types) which are solvable. One of these 28 models, in which the particles have exchange-interaction: $A + B \to B + A$, has been generalized to p species in reference [24]. In this model, besides diffusion to the right (with equal rates), particles interact through $A_j + A_i \to A_i + A_j$ with rate r_{ij}. The spectrum of r_{ij}, to ensure the solvability, has been obtained. Another multi-species reaction-diffusion model, which somehow relates to one considered in [24], has been discussed in [25]. In this model the particles have exchange-interaction with rates which are determined by the differences of their diffusion rates.

The multi-species generalization of the models considered in [23], has been considered in [26]. The processes are

$$\begin{aligned} A_\alpha \emptyset &\to \emptyset A_\alpha \quad \text{with rate } 1, \\ A_\alpha A_\beta &\to A_\gamma A_\delta \quad \text{with rate } c_{\gamma\delta}^{\alpha\beta}. \end{aligned} \tag{1}$$

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Again the solvability restricts the reaction rates $c_{\gamma\delta}^{\alpha\beta}$. Some general remarks on the solution of SYB equation, and some special solutions of it have been given in [26].

Recently, the generalization of the drop-push model to multi-species case has been considered in reference [27]. The reactions are

$$\begin{aligned} A_\alpha\emptyset &\to \emptyset A_\alpha \quad \text{with rate } 1, \\ A_\alpha A_\beta\emptyset &\to \emptyset A_\gamma A_\delta \quad \text{with rate } b_{\gamma\delta}^{\alpha\beta}. \end{aligned} \tag{2}$$

The same drop-push reactions, but with n adjacent particles and with rates that are specified by a specific combinations of $b_{\gamma\delta}^{\alpha\beta}$'s, also exist. Some comments have been given for solutions of SYB equation in [27]. The reactions (2) are called the extended drop-push reactions.

In this paper we are going to study the most general multi-species model, i.e. the most general boundary condition, which *all* the previous mentioned models are the special cases of it. In its general form, the reactions are

$$\begin{aligned} A_\alpha\emptyset &\to \emptyset A_\alpha \quad \text{with rate } 1, \\ A_\alpha A_\beta &\to A_\gamma A_\delta \quad \text{with rate } c_{\gamma\delta}^{\alpha\beta}, \\ A_\alpha A_\beta\emptyset &\to \emptyset A_\gamma A_\delta \quad \text{with rate } b_{\gamma\delta}^{\alpha\beta}, \\ &\vdots \end{aligned} \tag{3}$$

where the dots indicates the other drop-push reactions with n adjacent particles, in which in the meantime the types of the particles can also be changed. We show that the reaction rates must satisfy some specific constraints, in order that we have a set of consistent evolution equations. The two-particle S-matrices of this general multi-species model must also satisfy SYB equation, which in this case becomes more complicated. In the two-species case, we study a special class of reactions (3) in detail and show that in this class, there is only *one* solvable model, i.e. the solution of SYB equation is unique in this case. We study some physical properties of this unique model.

The scheme of the paper is the following. In Section 2, by using the law of conservation of probabilities, we obtain the most general boundary condition for a p-species reaction-diffusion model, in terms of two $p^2 \times p^2$ matrices b and c. We obtain a constraint on the sum of the elements of each column of matrix $b + c$. We see that in special cases, this model is same as those have been studied previously. In Section 3, we consider the matrix c as a diagonal matrix. It is shown that there are two cases. In the first case, the matrix b must be also diagonal, which the model reduces to the ordinary, i.e. single-species, drop-push model with variable rate. In the second case, c must be zero matrix, which the model becomes the one considered in [27], that is the extended drop-push model. In Section 4, we take c a non-diagonal matrix and obtain the necessary and sufficient conditions (constraints) for the matrices b and c, to have a consistent evolution equations. We see that the reactions are those indicated in equation (3). We find some classes of solutions of this set of constraints. In Section 5, we investigate the coordinate Bethe ansatz solution for our problem and obtain the SYB equation, to ensure the solvability of the model. For $p = 2$, we show that for a specific class of parameters, which corresponds to a special exchange-reaction together with a special extended drop-push model, the SYB equation has a unique solution. Finally in Section 6, the conditional two-particle probabilities are calculated for this unique model, and their large-time behaviour are studied.

2 The general boundary condition

Consider a p-species system with particles $A_1, A_2, \cdots, A_p$. The basic quantities that we are interested in are the probabilities $P_{\alpha_1\cdots\alpha_N}(x_1, \cdots, x_N; t)$ for finding at time t the particle of type α_1 at site x_1, particle of type α_2 at site x_2, etc. We take these functions to define probabilities only in the physical region $x_1 < x_2 < ... < x_N$. The most general master equation for an asymmetric exclusion process is

$$\begin{aligned} \frac{\partial}{\partial t}P_{\alpha_1\cdots\alpha_N}(x_1, \cdots, x_N; t) &= P_{\alpha_1\cdots\alpha_N}(x_1-1, \cdots, x_N; t)+\cdots \\ &+ P_{\alpha_1\cdots\alpha_N}(x_1, \cdots, x_N-1; t)-NP_{\alpha_1\cdots\alpha_N}(x_1, \cdots, x_N; t). \end{aligned} \tag{4}$$

This equation describes a collection of N particles drifting to the right with unit rate. It can be shown that if one decides to solve the master equation by coordinate Bethe ansatz method, one may only choose all the diffusion rates equal, which can be scaled to one [24]. So in equation (4), we take all the diffusion rates equal to one. The master equation (4) is only valid for

$$x_i < x_{i+1} - 1, \tag{5}$$

since for $x_i = x_{i+1} - 1$, there will be terms with $x_i = x_{i+1}$ in the right-hand side of equation (4), which is out of the physical region. One can, however, assume that (4) is correct for all the physical region $x_i < x_{i+1}$, and impose certain boundary conditions for $x_i = x_{i+1}$. These boundary conditions determine the nature of the interactions between particles. Now the question is that what are the possible boundary conditions? To see this, we follow the same argument which has been given for single-species model in [28]. If one considers the master equations (4) for two-particles probabilities, finds

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{x_2}\sum_{x_1<x_2} P_{\alpha_1\alpha_2}(x_1, x_2; t) &= \sum_{x_2}\sum_{x_1<x_2} [P_{\alpha_1\alpha_2}(x_1-1, x_2; t) \\ &+P_{\alpha_1\alpha_2}(x_1, x_2-1; t) - 2P_{\alpha_1\alpha_2}(x_1, x_2; t)] \\ &= \sum_{x_2}\sum_{x_1<x_2} P_{\alpha_1\alpha_2}(x_1, x_2; t) - \sum_x P_{\alpha_1\alpha_2}(x, x+1; t) \\ &+ \sum_{x_2}\sum_{x_1<x_2} P_{\alpha_1\alpha_2}(x_1, x_2; t) \\ &+ \sum_x P_{\alpha_1\alpha_2}(x, x; t) - 2\sum_{x_2}\sum_{x_1<x_2} P_{\alpha_1\alpha_2}(x_1, x_2; t) \\ &= -\sum_x P_{\alpha_1\alpha_2}(x, x+1; t) + \sum_x P_{\alpha_1\alpha_2}(x, x; t). \end{aligned} \tag{6}$$

Now let us exclude the creation and annihilation processes, in other words we consider the processes in which the number of particles is constant in time. Therefore if we sum equation (6) over α_1 and α_2, the left-hand side becomes zero, so the right-hand side must be also zero. The only possible choice that leads the right-hand side of equation (6) to zero, and consistent with more-than-two particle analogue of equation (6) (see Ref. [28]) is taking $P_{\alpha_1\alpha_2}(x,x;t)$ as a linear combination of $P_{\beta_1\beta_2}(x,x+1;t)$ and $P_{\beta_1\beta_2}(x-1,x;t)$'s. Therefore the most general boundary condition is

$$P_{\alpha_1\alpha_2}(x,x) = \sum_{\beta} b^{\beta_1\beta_2}_{\alpha_1\alpha_2} P_{\beta_1\beta_2}(x-1,x) + \sum_{\beta} c^{\beta_1\beta_2}_{\alpha_1\alpha_2} P_{\beta_1\beta_2}(x,x+1). \quad (7)$$

β stands for $(\beta_1\beta_2)$. These b and c matrices introduce interactions to particles. Inserting equation (7) in the right-hand side of equation (6) and sum over α_1 and α_2, results

$$-\sum_x \sum_\alpha P_{\alpha_1\alpha_2}(x,x+1) + \sum_x \sum_\beta \left(\sum_\alpha (b+c)^{\beta_1\beta_2}_{\alpha_1\alpha_2} \right) P_{\beta_1\beta_2}(x,x+1) = 0. \quad (8)$$

Therefore the sum over the elements of each column of the matrix $b+c$ must be equal to one,

$$\sum_\alpha (b+c)^{\beta_1\beta_2}_{\alpha_1\alpha_2} = 1. \quad (9)$$

Note that for annihilation process, the condition (9) need not be satisfied [28].

If the matrices b and c are diagonal, then the boundary condition (7) does not induce reactions in which the types of the particles are changed. So the left-hand side of (6) is zero, without summing over α_1 and α_2, and therefore

$$-\sum_x P_{\alpha_1\alpha_2}(x,x+1) + \sum_x (b+c)^{\alpha_1\alpha_2}_{\alpha_1\alpha_2} P_{\alpha_1\alpha_2}(x,x+1) = 0. \quad (10)$$

So in diagonal case, we have, instead of condition (9),

$$b = 1 - c. \quad (11)$$

The simple exclusion processes of [16] is an example of this model with $p=1$ and $b=0$, the drop-push model with equal rate is $p=1$ and $c=0$ case, and with non-equal rate, is an example of one-species case of equations (7) and (11) [17]. The two-species model of [23], special p-species of [24] and general p-species reactions of [26] are examples of equations (7) and (9) with $b=0$. Finally the extended drop-push model of [27] is an example with $c=0$. Now we are going to consider the situations which have not been studied yet, that is the multi-species boundary conditions in which both b and c matrices present.

3 Diagonal c

As the first step of our study, let us consider the matrix c to be diagonal. As we will show in the next section, the non-diagonal elements of c are reaction rates and so they are non-negative, but the diagonal elements are not reaction rates and are defined through equation (9), which for diagonal c results

$$c^{\beta_1\beta_2}_{\beta_1\beta_2} = 1 - \sum_\alpha b^{\beta_1\beta_2}_{\alpha_1\alpha_2}. \quad (12)$$

Now consider $\dot{P}_{\alpha_1\alpha_2}(x,x+1)$. Using equations (4), (7) and (12), one finds

$$\dot{P}_{\alpha_1\alpha_2}(x,x+1) = P_{\alpha_1\alpha_2}(x-1,x+1) + \sum_\beta b^{\beta_1\beta_2}_{\alpha_1\alpha_2} P_{\beta_1\beta_2}(x-1,x) - \left(1+\sum_\beta b^{\alpha_1\alpha_2}_{\beta_1\beta_2}\right) P_{\alpha_1\alpha_2}(x,x+1). \quad (13)$$

The above evolution equation determines the following reactions as the source and sink of this model

$$A_\alpha \emptyset \to \emptyset A_\alpha \quad \text{with rate } 1,$$
$$A_{\beta_1} A_{\beta_2} \emptyset \to \emptyset A_{\alpha_1} A_{\alpha_2} \quad \text{with rate } b^{\beta_1\beta_2}_{\alpha_1\alpha_2}. \quad (14)$$

To check the consistency of our description, we next consider the 3-particle probability $P_{\alpha_1\alpha_2\alpha_3}(x-1,x,x+1)$. In this case, the boundary term $P_{\alpha_1\alpha_2\alpha_3}(x-1,x,x)$ appears in the right-hand side of master equation. To calculate this term, we must use the boundary condition (7) in two steps. The result is

$$P_{\alpha_1\alpha_2\alpha_3}(x-1,x,x) = \sum_\beta b^{\beta_2\beta_3}_{\alpha_2\alpha_3} \left[\sum_\gamma b^{\gamma_1\gamma_2}_{\alpha_1\beta_2} P_{\gamma_1\gamma_2\beta_3}(x-2,x-1,x) + c^{\alpha_1\beta_2}_{\alpha_1\beta_2} P_{\alpha_1\beta_2\beta_3}(x-1,x,x) \right] + c^{\alpha_2\alpha_3}_{\alpha_2\alpha_3} P_{\alpha_1\alpha_2\alpha_3}(x-1,x,x+1). \quad (15)$$

It is seen that the boundary term $P_{\alpha_1\alpha_2\alpha_3}(x-1,x,x)$ is found as a linear combination of other boundary terms, i.e. $P_{\alpha_1\beta_2\beta_3}(x-1,x,x)$'s. To avoid this problem, which in general does not lead to a consistent description of the interactions, we have two choices. One choice is to restrict ourselves to the cases in which the multiplication factors of the boundary terms in the right-hand side of equation (15) are zero. Then

$$c^{\alpha_1\beta_2}_{\alpha_1\beta_2} b^{\beta_2\beta_3}_{\alpha_2\alpha_3} = \left(1 - \sum_\gamma b^{\alpha_1\beta_2}_{\gamma_1\gamma_2}\right) b^{\beta_2\beta_3}_{\alpha_2\alpha_3} = 0. \quad (16)$$

So for non-zero reaction rates, i.e. $b^{\beta_2\beta_3}_{\alpha_2\alpha_3} \neq 0$, we have $c^{\alpha_1\beta_2}_{\alpha_1\beta_2} = 0$ (i.e. $\sum_\gamma b^{\alpha_1\beta_2}_{\gamma_1\gamma_2} = 1$). In this case the model reduced to one studied in [27], that is the extended drop-push model.

The second choice is to consider the situations in which the boundary term in the right-hand side of equation (15) is the same as one appeared in the left-hand side. In other words, we choose

$$c^{\alpha_1\beta_2}_{\alpha_1\beta_2} b^{\beta_2\beta_3}_{\alpha_2\alpha_3} = r_{\alpha_1\alpha_2\alpha_3}\delta^{\beta_2}_{\alpha_2}\delta^{\beta_3}_{\alpha_3}. \tag{17}$$

This means that the matrix b must be also diagonal. Therefore we have

$$b^{\alpha\beta}_{\alpha\beta} := r_{\alpha\beta}\ ,\ c^{\alpha\beta}_{\alpha\beta} = 1 - r_{\alpha\beta}$$

$$P_{\alpha\beta}(x,x) = r_{\alpha\beta}P_{\alpha\beta}(x-1,x) + (1-r_{\alpha\beta})P_{\alpha\beta}(x,x+1). \tag{18}$$

Considering $\dot{P}_{\alpha\beta}(x, x+1)$, equations (4) and (18) show that the reactions are

$$A_\alpha\emptyset \to \emptyset A_\alpha \quad \text{with rate } 1,$$

$$A_\alpha A_\beta\emptyset \to \emptyset A_\alpha A_\beta \quad \text{with rate } r_{\alpha\beta}. \tag{19}$$

But if one considers $\dot{P}_{\alpha\beta\gamma}(x, x+1, x+2)$, or other n-adjacent particles probabilities, finds that the master equation (4), boundary condition (7), with b and c given in (18) can consistently describe the drop-push reactions only if $r_{\alpha\beta}$ is independent of α and β. In this way we arrive at the one-parameter drop-push model, previously studied in [17].

So in brief, considering c as a diagonal matrix, does not lead to any new model.

4 Non-diagonal c

Like the previous section, to find the reactions which exist in this case, we must first consider $\dot{P}_{\alpha_1\alpha_2}(x, x+1)$. Using equations (4), (7) and (9), we arrive at

$$\begin{aligned}\dot{P}_{\alpha_1\alpha_2}(x,x+1) &= P_{\alpha_1\alpha_2}(x-1,x+1) + \sum_\beta b^{\beta_1\beta_2}_{\alpha_1\alpha_2}P_{\beta_1\beta_2}(x-1,x)\\ &\quad + \sum_\beta c^{\beta_1\beta_2}_{\alpha_1\alpha_2}P_{\beta_1\beta_2}(x,x+1) - 2P_{\alpha_1\alpha_2}(x,x+1)\\ &= P_{\alpha_1\alpha_2}(x-1,x+1) + \sum_\beta b^{\beta_1\beta_2}_{\alpha_1\alpha_2}P_{\beta_1\beta_2}(x-1,x)\\ &\quad + \sum_{\beta\neq\alpha} c^{\beta_1\beta_2}_{\alpha_1\alpha_2}P_{\beta_1\beta_2}(x,x+1)\\ &\quad - \left(1 + \sum_\beta b^{\alpha_1\alpha_2}_{\beta_1\beta_2} + \sum_{\beta\neq\alpha} c^{\alpha_1\alpha_2}_{\beta_1\beta_2}\right)\\ &\quad \times P_{\alpha_1\alpha_2}(x,x+1),\end{aligned} \tag{20}$$

in which we use

$$c^{\alpha_1\alpha_2}_{\alpha_1\alpha_2} = 1 - \sum_\beta b^{\alpha_1\alpha_2}_{\beta_1\beta_2} - \sum_{\beta\neq\alpha} c^{\alpha_1\alpha_2}_{\beta_1\beta_2}. \tag{21}$$

The evolution equation (20) describes the following two-particle reactions:

$$\begin{aligned} A_\alpha\emptyset &\to \emptyset A_\alpha \quad \text{with rate } 1,\\ A_\alpha A_\beta &\to A_\gamma A_\delta \quad \text{with rate } c^{\alpha\beta}_{\gamma\delta},\\ A_\alpha A_\beta\emptyset &\to \emptyset A_\gamma A_\delta \quad \text{with rate } b^{\alpha\beta}_{\gamma\delta}.\end{aligned} \tag{22}$$

To find the more-than-two particles reactions, we must consider $\dot{P}_{\alpha_1\cdots\alpha_n}(x, x+1, \cdots, x+n-1)$. In $n=3$, we first need to know $P_{\alpha_1\alpha_2\alpha_3}(x-1, x, x)$. Using (7), we have

$$\begin{aligned}P_{\alpha_1\alpha_2\alpha_3}(x-1,x,x) &= \sum_{\beta\gamma} b^{\beta_2\beta_3}_{\alpha_2\alpha_3}\Big[b^{\gamma_1\gamma_2}_{\alpha_1\beta_2}P_{\gamma_1\gamma_2\beta_3}(x-2,x-1,x)\\ &\quad + c^{\gamma_1\gamma_2}_{\alpha_1\beta_2}P_{\gamma_1\gamma_2\beta_3}(x-1,x,x)\Big]\\ &\quad + \sum_\beta c^{\beta_2\beta_3}_{\alpha_2\alpha_3}P_{\alpha_1\beta_2\beta_3}(x-1,x,x+1).\end{aligned} \tag{23}$$

Again we consider two cases. In the first case, we take the matrices b and c such that

$$\sum_{\beta_2} c^{\gamma_1\gamma_2}_{\alpha_1\beta_2} b^{\beta_2\beta_3}_{\alpha_2\alpha_3} = r_{\alpha_1\alpha_2\alpha_3}\delta^{\gamma_1}_{\alpha_1}\delta^{\gamma_2}_{\alpha_2}\delta^{\beta_3}_{\alpha_3}. \tag{24}$$

Then equation (23) results

$$\begin{aligned}P_{\vec{\alpha}}(x-1,x,x) &= \sum_{\beta\gamma}\frac{b^{\beta_1\beta_2}_{\alpha_1\gamma}b^{\gamma\beta_3}_{\alpha_2\alpha_3}}{1-r_{\vec{\alpha}}}P_{\vec{\beta}}(x-2,x-1,x)\\ &\quad + \sum_\beta \frac{c^{\beta_2\beta_3}_{\alpha_2\alpha_3}}{1-r_{\vec{\alpha}}}P_{\alpha_1\beta_2\beta_3}(x-1,x,x+1),\end{aligned} \tag{25}$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \cdots)$. Using (25) in computing $\dot{P}_{\vec{\alpha}}(x-1, x, x+1)$, one can easily see that the resulting evolution equation gives different rates for same reaction. For example it gives the rate $c^{\beta_1\beta_2}_{\alpha_1\alpha_2}$ for process $\beta_1\beta_2\alpha_3 \to \alpha_1\alpha_2\alpha_3$, which is consistent with second reaction of equation (22), and rate $c^{\beta_2\beta_3}_{\alpha_2\alpha_3}/(1-r_{\vec{\alpha}})$ for reaction $\alpha_1\beta_2\beta_3 \to \alpha_1\alpha_2\alpha_3$. The only way to obtain the consistent result is taking $r_{\vec{\alpha}} = 0$. Therefore we restrict ourselves to the matrices b and c with property

$$\sum_{\beta_2} c^{\gamma_1\gamma_2}_{\alpha_1\beta_2} b^{\beta_2\beta_3}_{\alpha_2\alpha_3} = 0. \tag{26}$$

Note that this constraint is equivalent to

$$(1\otimes b)(c\otimes 1) = 0, \tag{27}$$

in which b and c are $p^2 \times p^2$ matrices satisfying (9), and 1 denotes the $p \times p$ identity matrix. Assuming (26) and using equations (4), (7) and (23), $\dot{P}_{\vec{\alpha}}(x, x+1, x+2)$ becomes

$$\begin{aligned}\dot{P}_{\vec{\alpha}}(x,x+1,x+2) = & P_{\vec{\alpha}}(x-1,x+1,x+2) \\ & + \sum_{\beta} b^{\beta_1\beta_2}_{\alpha_1\alpha_2} P_{\beta_1\beta_2\alpha_3}(x-1,x,x+2) \\ & + \sum_{\beta\neq\alpha} c^{\beta_1\beta_2}_{\alpha_1\alpha_2} P_{\beta_1\beta_2\alpha_3}(x,x+1,x+2) \\ & + \sum_{\beta} b^{\vec{\beta}}_{\vec{\alpha}} P_{\vec{\beta}}(x-1,x,x+1) \\ & + \sum_{\beta\neq\alpha} c^{\beta_2\beta_3}_{\alpha_2\alpha_3} P_{\alpha_1\beta_2\beta_3}(x,x+1,x+2) \\ & - \Bigg(1 + \sum_{\beta} b^{\alpha_1\alpha_2}_{\beta_1\beta_2} + \sum_{\beta\neq\alpha} c^{\alpha_1\alpha_2}_{\beta_1\beta_2} \\ & + \sum_{\beta} b^{\alpha_2\alpha_3}_{\beta_2\beta_3} + \sum_{\beta\neq\alpha} c^{\alpha_2\alpha_3}_{\beta_2\beta_3} \Bigg) \\ & \times P_{\vec{\alpha}}(x,x+1,x+2), \end{aligned} \tag{28}$$

in which we use equation (21) for $c^{\alpha_1\alpha_2}_{\alpha_1\alpha_2}$ and $c^{\alpha_2\alpha_3}_{\alpha_2\alpha_3}$. In above equation, $b^{\vec{\beta}}_{\vec{\alpha}}$ stands for

$$b^{\vec{\beta}}_{\vec{\alpha}} = \sum_{\gamma} b^{\beta_1\beta_2}_{\alpha_1\gamma} b^{\gamma\beta_3}_{\alpha_2\alpha_3}. \tag{29}$$

The source terms of equation (28), besides the reactions (22), are the following 3-particle drop-push reaction

$$A_{\beta_1} A_{\beta_2} A_{\beta_3} \emptyset \to \emptyset A_{\alpha_1} A_{\alpha_2} A_{\alpha_3} \quad \text{with rate } b^{\vec{\beta}}_{\vec{\alpha}}. \tag{30}$$

The sink terms are also consistent with reactions (22) and (30), provided

$$\sum_{\beta} b^{\vec{\alpha}}_{\vec{\beta}} = \sum_{\beta\gamma} b^{\alpha_1\alpha_2}_{\beta_1\gamma} b^{\gamma\alpha_3}_{\beta_2\beta_3} = \sum_{\beta} b^{\alpha_1\alpha_2}_{\beta_1\beta_2}. \tag{31}$$

This is the last constraints that must be satisfied by the elements of matrix b. It can be shown that the more-than-three adjacent particles probabilities are consistent with following reactions

$$\begin{aligned} A_\alpha \emptyset \to \emptyset A_\alpha \ & \text{ with rate } \ 1, \\ A_\alpha A_\beta \to A_\gamma A_\delta \ & \text{ with rate } \ c^{\alpha\beta}_{\gamma\delta}, \\ A_{\alpha_0}\cdots A_{\alpha_n}\emptyset \to \emptyset A_{\gamma_0}\cdots A_{\gamma_n} & \\ & \text{with rate } (b_{n-1,n}\cdots b_{0,1})^{\alpha_0\cdots\alpha_n}_{\gamma_0\cdots\gamma_n}, \end{aligned} \tag{32}$$

if the constraints (9), (26) and (31) are satisfied. In equation (32), we use the following definition

$$b_{k,k+1} = 1\otimes\cdots\otimes 1\otimes \underbrace{b}_{k,k+1} \otimes 1\otimes\cdots\otimes 1. \tag{33}$$

The important point is that we need not any further constraint. For example in describing the source and sink terms of the evolution equation of $P_{\vec{\alpha}}(x, x+1, x+2, x+3)$, we encounter the constraint

$$\sum_{\beta\gamma\theta} b^{\alpha_1\alpha_2}_{\beta_1\gamma} b^{\gamma\alpha_3}_{\beta_2\theta} b^{\theta\alpha_4}_{\beta_3\beta_4} = \sum_{\beta} b^{\alpha_1\alpha_2}_{\beta_1\beta_2}. \tag{34}$$

Now as (see Eq. (31))

$$\sum_{\beta\theta} b^{\gamma\alpha_3}_{\beta_2\theta} b^{\theta\alpha_4}_{\beta_3\beta_4} = \sum_{\beta} b^{\gamma\alpha_3}_{\beta_2\beta_3}, \tag{35}$$

constraint (34) reduces to $\sum_{(\beta\gamma)} b^{\alpha_1\alpha_2}_{\beta_1\gamma} b^{\gamma\alpha_3}_{\beta_2\beta_3} = \sum_{\beta} b^{\alpha_1\alpha_2}_{\beta_1\beta_2}$, which is nothing but equation (31). This completes our investigation and the final processes are those indicated in (32). Of course, the question of solvability is open yet.

As the elements of matrix b and the non-diagonal elements of matrix c are reaction rates, they must be non-negative, and this point restricts the allowed solutions of equations (9), (26) and (31). Let us focus on two class of solutions. To be specific, we consider $p = 2$ case, but the arguments can be easily applied to arbitrary p. In $p = 2$, we may denote the two-particle states (α_1, α_2) as following:

$$|1\rangle = (1,1),\ |2\rangle = (1,2),\ |3\rangle = (2,1),\ |4\rangle = (2,2). \tag{36}$$

solution of class 1: If $b^{\beta_2\beta_3}_{\alpha_2\alpha_3}$ in equation (26) is independent of β_2, that is

$$b^{1\beta_3}_{\alpha_2\alpha_3} = b^{2\beta_3}_{\alpha_2\alpha_3} = \cdots = b^{p\beta_3}_{\alpha_2\alpha_3}, \tag{37}$$

then equation (26) reduces to

$$\sum_{\beta} c^{\gamma_1\gamma_2}_{\alpha_1\beta} = 0. \tag{38}$$

For $p = 2$, equation (37) gives ${b_i}^1 = {b_i}^3$ (for $\beta_3 = 1$) and ${b_i}^2 = {b_i}^4$ (for $\beta_3 = 2$), and equation (38) for $\alpha_1 = 1, 2$ specifies the elements of c as following

$$c = \begin{pmatrix} -{c_2}^1 & {c_1}^2 & 0 & 0 \\ {c_2}^1 & -{c_1}^2 & 0 & 0 \\ 0 & 0 & -{c_4}^3 & {c_3}^4 \\ 0 & 0 & {c_4}^3 & -{c_3}^4 \end{pmatrix}. \tag{39}$$

It can be easily shown that if we take the sum of elements of each column of b equal to one, then the constraints (9) and (31) are also satisfied. The resulting model is a ten-parameter reaction-diffusion model.

solution of class 2: If we take, for each β, one or both of $c^{\gamma_1\gamma_2}_{\alpha_1\beta}$ and $b^{\beta\gamma_3}_{\alpha_2\alpha_3}$ equal to zero, then equation (26) is satisfied. In $p = 2$, we can take $c^{\gamma_1\gamma_2}_{\alpha_1 2} = 0$ and $b^{1\gamma_3}_{\alpha_2\alpha_3} = 0$, which means that taking zero the second and fourth rows of c and first and second columns of b. Then solving (31),

results ${b_2}^3 + {b_4}^3 = {b_2}^4 + {b_4}^4 = 1$ and $\sum_i {b_i}^3 = \sum_i {b_i}^4$. Finally considering (9) and noting the non-negativity of reaction rates, gives

$$b = \begin{pmatrix} 0\,0 & 0 & 0 \\ 0\,0 & {b_2}^3 & {b_2}^4 \\ 0\,0 & 0 & 0 \\ 0\,0\,1-{b_2}^3 & 1-{b_2}^4 & \end{pmatrix},$$

$$c = \begin{pmatrix} 1-{c_3}^1 & {c_1}^2 & {c_1}^3 & 0 \\ 0 & 0 & 0 & 0 \\ {c_3}^1 & 1-{c_1}^2 & -{c_1}^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (40)$$

with condition

$$\left({b_2}^3, {b_2}^4, {c_1}^2\right) \leq 1. \quad (41)$$

In $p = 2$, one can show that these two classes are the complete set of solutions of the constraint equations. Now we must seek the Bethe ansatz solution for the allowed set of parameters.

5 The Bethe ansatz solution

To solve the master equation (4) with boundary condition (7), we consider the following ansatz

$$P_{\alpha_1,\cdots,\alpha_N}(\mathbf{x};t) = e^{-E_N t}\psi_{\alpha_1,\cdots,\alpha_N}(\mathbf{x}), \quad (42)$$

with

$$\Psi(\mathbf{x}) = \sum_\sigma \mathbf{A}_\sigma e^{i\sigma(\mathbf{p}).\mathbf{x}}. \quad (43)$$

Ψ is a tensor of rank N with components $\psi_{\alpha_1,\cdots,\alpha_N}(\mathbf{x})$, and the summation runs over the elements of the permutation group of N objects [29,30]. Inserting (42) in equations (4) and (7), results, respectively,

$$E_N = \sum_{k=1}^{N} \left(1 - e^{-ip_k}\right), \quad (44)$$

and

$$\Psi(\cdots, x_k = x, x_{k+1} = x, \cdots) = b_{k,k+1}\Psi(\cdots, x_k = x-1, x_{k+1} = x, \cdots) + c_{k,k+1}\Psi(\cdots, x_k = x, x_{k+1} = x+1, \cdots). \quad (45)$$

$c_{k,k+1}$ defined like $b_{k,k+1}$ in equation (33). The coefficients $\mathbf{A}_\sigma$ can be determined by inserting equation (43) in (45), which gives

$$\left[1 - e^{-i\sigma(p_k)}b_{k,k+1} - e^{i\sigma(p_{k+1})}c_{k,k+1}\right]\mathbf{A}_\sigma + \left[1 - e^{-i\sigma(p_{k+1})}b_{k,k+1} - e^{i\sigma(p_k)}c_{k,k+1}\right]\mathbf{A}_{\sigma\sigma_k} = 0. \quad (46)$$

σ_k is an element of permutation group which only interchanges p_k and p_{k+1}:

$$\sigma_k : (p_1, \cdots, p_k, p_{k+1}, \cdots, p_N) \rightarrow (p_1, \cdots, p_{k+1}, p_k, \cdots, p_N). \quad (47)$$

Using (46), $\mathbf{A}_{\sigma\sigma_k}$ is obtained from $\mathbf{A}_\sigma$ as following

$$\mathbf{A}_{\sigma\sigma_k} = S_{k,k+1}(\sigma(p_k), \sigma(p_{k+1}))\mathbf{A}_\sigma, \quad (48)$$

where

$$S_{k,k+1}(z_1, z_2) = 1 \otimes \cdots \otimes 1 \otimes \underbrace{S(z_1, z_2)}_{k,k+1} \otimes 1 \otimes \cdots \otimes 1. \quad (49)$$

$S(z_1, z_2)$ is the following $p^2 \times p^2$ matrix

$$S(z_1, z_2) = -\left(1 - z_2^{-1}b - z_1 c\right)^{-1}\left(1 - z_1^{-1}b - z_2 c\right), \quad (50)$$

and $z_k = e^{ip_k}$.

Until now, all the b and c matrices are acceptable. But we must note that the generators of the permutation group satisfy $\sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}$, so one also needs

$$\mathbf{A}_{\sigma_k\sigma_{k+1}\sigma_k} = \mathbf{A}_{\sigma_{k+1}\sigma_k\sigma_{k+1}}. \quad (51)$$

In terms of S-matrices, equation (51) becomes

$$S_{12}(z_2, z_3)S_{23}(z_1, z_3)S_{12}(z_1, z_2) = S_{23}(z_1, z_2)S_{12}(z_1, z_3)S_{23}(z_2, z_3). \quad (52)$$

Writing S-matrix as the product of the permutation matrix Π and a R matrix:

$$S_{k,k+1} =: \Pi_{k,k+1}R_{k,k+1}, \quad (53)$$

equation (52) is transformed to

$$R_{23}(z_2, z_3)R_{13}(z_1, z_3)R_{12}(z_1, z_2) = R_{12}(z_1, z_2)R_{13}(z_1, z_3)R_{23}(z_2, z_3). \quad (54)$$

This is the spectral Yang-Baxter equation [31–33].

Now the matrices b and c must be such that if one computes the two-particle S-matrix (50) by them, it satisfies the spectral Yang-Baxter equation (54), or equivalently equation (52). These equations are complicated and determining the elements of b and c, in such a way that equation (52) is satisfied, is not easy.

For $b = 0$, the S-matrix (50) becomes a binomial of degree one with respect to z_2 and therefore (52) becomes a quadratic expression with respect to z_3. Using this, one can transform the spectral Yang-Baxter equation (52) to a non-spectral matrix equation for matrix c, which its study is much easier than equation (52) [26]. The same is true for $c = 0$, since in this case equation (50) is linear in $z_1^{-1} = e^{-ip_1}$ [27]. When b and c are both different from zero, this procedure does not work because of the presence of z_2 and z_2^{-1} (or equally z_1 and z_1^{-1}) in both terms of (50).

For matrices b and c which are given in equation (40), if one computes the corresponding S-matrix and writes equation (52) as RHS – LHS = 0, then one has 64 equations that must be solved for five variables ${b_2}^3, {b_2}^4, {c_1}^2, {c_1}^3$ and ${c_3}^1$. The solutions must be momentum-independent and non-negative. By obtaining the set of solutions of this system of equations by standard mathematical softwares, it is seen that all of the solutions are momentum-dependent except one. This only acceptable solution

is: ${b_2}^3 = 1$, and other four parameters are zero. Therefore the solvable two-species reaction-diffusion model is defined through the following b and c matrices:

$$b = \begin{pmatrix} 0\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,0\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix},\ c = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,0 \end{pmatrix}. \tag{55}$$

These matrices introduce the following reactions, with $A \equiv A_1$ and $B \equiv A_2$,

$$\begin{aligned} A\emptyset &\to \emptyset A \\ B\emptyset &\to \emptyset B \\ AB &\to BA \\ BA\emptyset &\to \emptyset AB \\ BB\emptyset &\to \emptyset BB \\ BAA\emptyset &\to \emptyset AAB \\ BAB\emptyset &\to \emptyset ABB \\ BBA\emptyset &\to \emptyset BAB \\ BBB\emptyset &\to \emptyset BBB \\ &\vdots \end{aligned} \tag{56}$$

The dots indicates the more-than-three particle drop-push reactions which are specified by equations (32) and (33), and all reactions occur with rate one.

6 Two-particle conditional probabilities for reactions (56)

The conditional probability $P(\vec{\alpha}, \mathbf{x}; t|\vec{\beta}, \mathbf{y}; 0)$ is the probability of finding particles $\alpha_1, \alpha_2, \cdots$ at time t at sites $x_1, x_2, \cdots$, respectively, if at $t = 0$ we have particles $\beta_1, \beta_2, \cdots$ at sites $y_1, y_2, \cdots$, respectively. In two-particle sector, it is

$$\begin{pmatrix} P_{AA} \\ P_{AB} \\ P_{BA} \\ P_{BB} \end{pmatrix} (\mathbf{x}; t|\vec{\beta}, \mathbf{y}; 0) = \frac{1}{(2\pi)^2} \int e^{-E_2 t} e^{-i\mathbf{p}.\mathbf{y}} \Psi(x_1, x_2) dp_1 dp_2, \tag{57}$$

where by equations (43), (48), (50) and (55), $\Psi(x_1, x_2)$ is:

$$\Psi(x_1, x_2) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} e^{i\mathbf{p}.\mathbf{x}} + S_{12}(z_1, z_2) \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} e^{i\widetilde{\mathbf{p}}.\mathbf{x}}, \tag{58}$$

in which $\mathbf{p} = (p_1, p_2)$, $\widetilde{\mathbf{p}} = (p_2, p_1)$, and $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ stands for $\mathbf{A}_{\sigma=1}$ and specified by the initial condition. $z_1 = e^{ip_1}$, $z_2 = e^{ip_2}$, and $S_{12}(z_1, z_2)$ is:

$$S_{12}(z_1, z_2) = \begin{pmatrix} \frac{z_2 - 1}{1 - z_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{z_1} & 0 \\ 0 & z_2 & 0 & 0 \\ 0 & 0 & 0 & \frac{z_2(1 - z_1)}{z_1(z_2 - 1)} \end{pmatrix}. \tag{59}$$

After some calculations, one finds (see for example Refs. [23] and [24] for more details):

$$\begin{aligned} P(A, A, \mathbf{x}; t|A, A, \mathbf{y}; 0) &= F_1(t) + F_2(t), \\ P(A, B, \mathbf{x}; t|A, B, \mathbf{y}; 0) &= F_1(t), \\ P(B, A, \mathbf{x}; t|A, B, \mathbf{y}; 0) &= F_4(t), \\ P(B, A, \mathbf{x}; t|B, A, \mathbf{y}; 0) &= F_1(t), \\ P(A, B, \mathbf{x}; t|B, A, \mathbf{y}; 0) &= F_3(t), \\ P(B, B, \mathbf{x}; t|B, B, \mathbf{y}; 0) &= F_1(t) + F_5(t), \end{aligned} \tag{60}$$

and all other probabilities are zero. $F_i(t)$'s are:

$$\begin{aligned} F_1(t) &= e^{-2t} \frac{t^{x_1 - y_1}}{(x_1 - y_1)!} \frac{t^{x_2 - y_2}}{(x_2 - y_2)!}, \\ F_2(t) &= e^{-2t} \left\{ \frac{t^{x_1 - y_2 + 1}}{(x_1 - y_2 + 1)!} - \frac{t^{x_1 - y_2}}{(x_1 - y_2)!} \right\} \\ &\quad \times \sum_{k=0}^{\infty} \frac{t^{x_2 - y_1 + k}}{(x_2 - y_1 + k)!}, \\ F_3(t) &= e^{-2t} \frac{t^{x_2 - y_1 - 1}}{(x_2 - y_1 - 1)!} \frac{t^{x_1 - y_2}}{(x_1 - y_2)!}, \\ F_4(t) &= e^{-2t} \frac{t^{x_2 - y_1}}{(x_2 - y_1)!} \frac{t^{x_1 - y_2 + 1}}{(x_1 - y_2 + 1)!}, \\ F_5(t) &= e^{-2t} \left\{ \frac{t^{x_2 - y_1}}{(x_2 - y_1)!} - \frac{t^{x_2 - y_1 - 1}}{(x_2 - y_1 - 1)!} \right\} \\ &\quad \times \sum_{k=1}^{\infty} \frac{t^{x_1 - y_2 + k}}{(x_1 - y_2 + k)!}. \end{aligned} \tag{61}$$

Note that beginning with initial state (A, B), the system can go to state (B, A) by third reaction of (56) and again goes back to (A, B) by fourth reaction, etc.. As the rates of reactions are equal, we expect that at large time, the probabilities of finding (A, B) and (B, A) are equal, if they are summed over all accessible sites. In fact, if one calculates the difference of these two probabilities, finds:

$$\begin{aligned} D(t) =: & \sum_{x_2 = y_2}^{\infty} \sum_{x_1 = y_1}^{x_2 - 1} [P(A, B, \mathbf{x}; t|A, B, \mathbf{y}; 0) \\ & - P(B, A, \mathbf{x}; t|A, B, \mathbf{y}; 0)] \\ = & \ e^{-2t} \left[2 \sum_{m=1}^{y_2 - y_1 - 1} I_m(2t) + I_0(2t) \right], \end{aligned} \tag{62}$$

in which $I_n(x)$ is the n-th order Bessel function of the first kind. Now at $x \to \infty$, we have

$$I_n(x) \to \frac{e^x}{\sqrt{2\pi x}}, \tag{63}$$

so

$$\lim_{t \to \infty} D(t) = \frac{M}{\sqrt{4\pi t}} \to 0, \tag{64}$$

where M is the number of $I_n(2t)$s in equation (62). The final result confirms our expectation. The same result can be proved when the initial state is (B, A).

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